

Fitzpatrick Algorithm for Multivariate Rational Interpolation

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Abstract

In this paper, we first apply the Fitzpatrick algorithm to osculatory rational interpolation. Then based on Fitzpatrick algorithm, we present a Neville-like algorithm for Cauchy interpolation. With this algorithm, we can determine the value of the interpolating function at a single point without computing the rational interpolating function.

Key words: Fitzpatrick algorithm, Rational interpolation, Gröbner basis, Neville-like algorithm

1. Introduction

Interpolation is an important method in numerical approximation. Rational functions sometimes are superior to polynomial for interpolating data because they can achieve more accurate approximations with the same amount of computation [1]. In addition, rational interpolants have a natural way of interpolating poles whereas polynomial interpolants do not. So how to solve the problem of rational interpolation is what people have been concerning.

There are rich literatures on the univariate Cauchy interpolation and osculatory rational interpolation problem, such as [2, 3, 4, 5, 6, 7, 8, 9]. For multivariate rational interpolation [2, 3, 4, 10, 11] gave some results about bivariate cases and the authors assume the interpolation nodes are

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on rectangular grids. [12] computed rational interpolation over pyramid-typed grids in \mathbb{R}^3 by branched continued fractions. When the interpolation data are scattered, [13] presented a fast solver for the linear block Cauchy-Vandermonde system that translates the interpolation conditions. [14] used the theory of algebraic geometry to study the minimal multivariate rational interpolation.

One of the main problems of rational interpolation is the parametrization of all solutions of a given degree of complexity. Based on Euclidean algorithm, [15] investigated a general frame work which lead to a parametrization of all rational interpolation functions. [16] considered the set $M = \{(a, b) : a \equiv bh \pmod{x^{2t}}\}$ of all solutions of the key equation for alternant codes, and give the Fitzpatrick algorithm. [17] extended the Fitzpatrick algorithm to determining a parametrization of all minimal complexity rational functions $a(x)/b(x)$ interpolating an arbitrary sequence of points, and complexity is measured in terms of $\max\{\deg(a(x)), \deg(b(x)) + \xi\}$, where ξ is a given integer. [18] presented an algorithm to seeking the Gröbner basis for the solution of polynomial congruences in one or more variables. [19] generalized the work in [18], and got a general algorithm applicable to a wide range of constrained interpolation.

In this paper we apply the Fitzpatrick algorithm which appears in [19] to multivariate osculatory rational interpolation, and get the parametric solution of the multivariate osculatory rational interpolation function $r(x) = a(X)/b(X)$. Based on Fitzpatrick algorithm, we present a Neville-like algorithm for multivariate Cauchy interpolation.

The rest of the paper is organized as follows. In Section 2, we present the Fitzpatrick algorithm. In Section 3, we apply the Fitzpatrick algorithm to seek the weak solution $(a(X), b(X))$ of multivariate osculatory rational interpolation. In Section 4, based on Fitzpatrick algorithm, we give a Neville-like algorithm for multivariate Cauchy interpolation. With this algorithm, we can determine the value of the interpolating function at a single point without computing the rational interpolating function.

2. Fitzpatrick algorithm

Fitzpatrick algorithm, also called FGLM-like algorithm, is applicable to coding theory, Padé approximation, partial realization, interpolation, and other fields. [20] described these literatures from a historical point of view, started from [16], and covered recent developments for list decoding. For

further details, please refer to [20] or [21]. Now we introduce the structure of Fitzpatrick algorithm.

Let \mathbb{F} be a field, $\mathcal{P} = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring and $d \geq 1$ be a natural number.

We denote by $\{\vec{e}_1, \dots, \vec{e}_d\}$ the canonical basis of \mathcal{P}^d . Any term in \mathcal{P}^d is of the form $m = \phi \vec{e}_k$, $1 \leq k \leq d$, where ϕ is a term in \mathcal{P} , and the set of terms in \mathcal{P}^d is denoted by $\mathcal{T}^{(d)}$. Let \prec be a term order. For each $\vec{f} = \sum_{\tau \in \mathcal{T}^{(d)}} c(\vec{f}, \tau) \tau \in \mathcal{P}^d$, its support is

$$\text{supp}(\vec{f}) := \{\tau \in \mathcal{T}^d : c(\vec{f}, \tau) \neq 0\},$$

its *leading term* is $\mathbf{LT}(\vec{f}) := \max_{\prec}(\text{supp}(\vec{f}))$, its *leading coefficient* is $\mathbf{LC}(\vec{f}) := c(\vec{f}, \mathbf{LT}(\vec{f}))$ and *leading monomial* is $\mathbf{LM}(\vec{f}) := \mathbf{LC}(\vec{f})\mathbf{LT}(\vec{f})$.

Let M_k, M_{k+1} be submodules of a \mathcal{P} -module M , with $M_k \supseteq M_{k+1}$, such that, for each s , $1 \leq s \leq n$, there exists $\beta_s \in \mathbb{F}$ satisfying

$$(x_s - \beta_s)M_k \subseteq M_{k+1}. \quad (1)$$

For each k , there exists an \mathbb{F} -homomorphism

$$\theta_k : M_k \longrightarrow \mathbb{F} \quad (2)$$

with $\ker(\theta_k) = M_{k+1}$.

In [19], they described Fitzpatrick algorithm in the following theorem.

Theorem 1. [19] *Let M be a \mathcal{P} -module and let $M_k \supseteq M_{k+1}$ be submodules of M satisfying (1) and (2) for suitable β_s, θ_k . Let $H : \mathcal{P}^d \longrightarrow M$ be an \mathbb{F} -linear function such that for each s , $1 \leq s \leq n$, there exists $\gamma_s \in \mathbb{F}$ satisfying*

$$H(x_s \vec{b}) = (x_s + \gamma_s)H(\vec{b})$$

for all $\vec{b} = (b_1, \dots, b_d) \in \mathcal{P}^d$. Let $S \subseteq \mathcal{P}^d$ be a submodule satisfying

$$H(\vec{b}) \equiv 0 \pmod{M_k}, \forall \vec{b} \in S \quad (3)$$

and let $S' \subseteq S$ be the set of elements satisfying

$$H(\vec{b}) \equiv 0 \pmod{M_{k+1}} \quad (4)$$

Then S' is a submodule of \mathcal{P}^d .

If we have obtained an ordered minimal Gröbner basis $G = \{G[1], \dots, G[|G|]\}$ of S with respect to a term order \prec , then a Gröbner basis G' of S' with respect to \prec can be constructed as follows:

Define $\alpha_j = \theta_k(H(G[j]))$, for $1 \leq j \leq |G|$.

If $\alpha_j = 0$ for all j then

$$G' = G$$

otherwise

$j^* =$ the least j for which $\alpha_j \neq 0$

$$G_1 := \{G[j] : j < j^*\}$$

$$G_2 := \{(x_s - (\beta_s + \gamma_s))G[j^*] : 1 \leq s \leq n\}$$

$$G_3 := \{G[j] - (\alpha_j/\alpha_{j^*})G[j^*] : j > j^*\}$$

$$G' := G_1 \cup G_2 \cup G_3.$$

In the following section we will use the Fitzpatrick algorithm to compute osculatory rational interpolation functions.

3. Osculatory rational interpolation and Fitzpatrick algorithm

Now, we introduce some notations. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We define a differential operator D^α by

$$D^\alpha = \frac{1}{\alpha_1! \dots \alpha_n!} \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \triangleq \frac{1}{\alpha!} \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

A subset $\mathcal{A} \subset \mathbb{N}^n$ is called a *lower set*, if it is closed under the division order, that is, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{A}$ then $\beta \in \mathcal{A}$ for all $\beta = (\beta_1, \dots, \beta_n)$ with $\beta_i \leq \alpha_i, i = 1, \dots, n$.

The multivariate osculatory rational interpolation problem can be stated as follows:

Given a set of L distinct points $\{Y_1, \dots, Y_L\}$ in space \mathbb{F}^n . Point Y_i has multiplicity defined by the lower set \mathcal{A}_i , and the corresponding values $\{f_i^{(\alpha)} \in \mathbb{F} : \forall \alpha \in \mathcal{A}_i, i = 1, \dots, L\}$. Construct a rational interpolation function

$$r(X) = \frac{a(X)}{b(X)},$$

such that

$$D^\alpha r(X)|_{Y_i} = f_i^{(\alpha)}, \quad \forall \alpha \in \mathcal{A}_i, \quad i = 1, \dots, L,$$

where $a \in \mathcal{P}$, $b \in \mathcal{P}$, $a \neq 0$, $b(Y_i) \neq 0$ for all i .

We call this problem multivariate osculatory rational interpolation, and $r(X)$ multivariate osculatory rational interpolation function.

3.1. Weak interpolation

From the definition of multivariate osculatory rational interpolation we know that the equivalent definition is: the Taylor series expansion of $r(X)$ at the point $X = Y_i$ satisfies

$$r(X) = \sum_{\alpha_{i,j} \in \mathcal{A}_i} (X - Y_i)^{\alpha_{i,j}} f_i^{(\alpha_{i,j})} + \dots$$

Let $s_i = \sharp \mathcal{A}_i$, $i = 1, \dots, L$, $N = \sum_{i=1}^L s_i$. For each point Y_i and the lower set \mathcal{A}_i , define polynomial h_i

$$h_i := \sum_{\alpha_{i,j} \in \mathcal{A}_i} (X - Y_i)^{\alpha_{i,j}} f_i^{(\alpha_{i,j})}.$$

For each \mathcal{A}_i , rearrange the elements of \mathcal{A}_i , such that each subset $\mathcal{A}_{i,j} = \{\alpha_{i,0}, \dots, \alpha_{i,j}\}$, $0 \leq j \leq s_i - 1$, is still a lower set. In particular, $\mathcal{A}_{i,s_i-1} = \mathcal{A}_i$.

Denote the ideal $I((Y_i, \mathcal{A}_{i,j})) = \{p \in \mathcal{P} : D^\alpha p(X)|_{Y_i} = 0, \forall \alpha \in \mathcal{A}_{i,j}\}$ by $I_{i,j}$ and call $I_{i,j}$ the *vanishing ideal* of $(Y_i, \mathcal{A}_{i,j})$.

Definition 1. (Weak interpolation) A pair $(a, b) \in \mathcal{P}^2$ is called a *weak interpolation* for multivariate osculatory rational interpolation problem if

$$a \equiv b h_i \pmod{I_{i,s_i-1}}, i = 1, \dots, L.$$

Define $(a, b) + (c, d) = (a + c, b + d)$, $d(a, b) = (da, db)$. Thus $M = \{(a, b) : a \equiv b h_i \pmod{I_{i,s_i-1}}, i = 1, \dots, L\}$ is a \mathcal{P} -submodule.

If $\{(a_1, b_1), \dots, (a_t, b_t)\}$ is a Gröbner basis of the submodule M , then any pair (a, b) with the form

$$(a, b) = c_1(a_1, b_1) + \dots + c_t(a_t, b_t)$$

is a weak interpolation, where $c_j \in \mathcal{P}$ ($j = 1, \dots, t$) are free parameters. Choose c_j properly such that $b(Y_i) \neq 0$, $i = 1, \dots, L$, then we can get the interpolation function

$$\frac{a(X)}{b(X)} = \frac{c_1 a_1 + \dots + c_t a_t}{c_1 b_1 + \dots + c_t b_t}.$$

3.2. Fitzpatrick algorithm_RI

The aim of this subsection is to apply the Fitzpatrick algorithm to compute osculatory rational interpolation.

Definition 2. (order \prec_ξ)

1. We say $X^\alpha(1, 0) \prec_\xi X^\beta(1, 0)$ if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and $X^\alpha \prec_{lex} X^\beta$,
2. We say $X^\alpha(1, 0) \prec_\xi X^\beta(0, 1)$ if $|\alpha| \leq |\beta| + \xi$,
3. We say $X^\alpha(0, 1) \prec_\xi X^\beta(0, 1)$ if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and $X^\alpha \prec_{lex} X^\beta$,

where \prec_{lex} is the lexicographic order on \mathcal{P} , and ξ is a given integer.

It is easy to check that the order \prec_ξ is a monomial order on \mathcal{P}^2 .

For each $\mathcal{A}_{i,j}$, $1 \leq i \leq L$, $0 \leq j \leq s_i - 1$, define the *congruent equation* as

$$a \equiv bh_i \pmod{I_{i,j}},$$

where $I_{i,j}$ is the vanishing ideal of $(Y_i, \mathcal{A}_{i,j})$.

Define an order $<$ on the lower sets $\{\mathcal{A}_{i,j}, 1 \leq i \leq L, 0 \leq j \leq s_i - 1\}$ such that $\mathcal{A}_{i_1,j_1} < \mathcal{A}_{i_2,j_2}$ if and only if $i_1 < i_2$, or $i_1 = i_2 = i$ and $\mathcal{A}_{i,j_1} \subset \mathcal{A}_{i,j_2}$ for $j_1 < j_2$.

Consequently, an order on the congruent equations is induced:

$$a \equiv bh_i \pmod{I_{i_1,j_1}} < a \equiv bh_i \pmod{I_{i_2,j_2}} \text{ if and only if } \mathcal{A}_{i_1,j_1} < \mathcal{A}_{i_2,j_2}.$$

Now we can establish a one to one correspondence between index k and (i, j) . Define a sequence of submodules M_k , $k = 0, \dots, N$, where $M_0 = \mathcal{P}^2$, M_k is the set of common solutions of the first k congruent equations

$$M_k = \{(a, b) \in M_{k-1} : a \equiv bh_{i_k} \pmod{I_{i_k,j_k}}, k = 1, \dots, N.$$

Obviously $M = M_N = \{(a, b) : a \equiv bh_i \pmod{I_{i,s_i-1}}, i = 1, \dots, L\}$, and $M_0 \supseteq M_1 \supseteq \dots \supseteq M_N$.

Fix an order \prec_ξ we compute the minimal Gröbner basis \mathcal{G}_N of M_N recursively. It is easy to see that $\{(1, 0), (0, 1)\}$ is a Gröbner basis of M_0 . We compute the Gröbner basis \mathcal{G}_{k+1} of M_{k+1} through the minimal Gröbner basis \mathcal{G}_k of M_k .

Let $\mathcal{G}_k = \{(a_1, b_1), \dots, (a_{m_k}, b_{m_k})\}$ be the minimal Gröbner basis of M_k , and the $(k+1)$ -th congruent equation be $a \equiv bh_l \pmod{I_{l,k_l}}$. Then

1. if $k_l = 0$, that is $\mathcal{A}_{l,k_l} = \mathcal{A}_{l,0} = \{\alpha_{l,0} = 0\}$, then

$$bh_l - a \equiv \nu \equiv \nu(X - Y_l)^{\alpha_{l,0}} \pmod{I_{l,0}}.$$

2. if $k_l \neq 0$, that is $\mathcal{A}_{l,k_l} \setminus \mathcal{A}_{l,k_l-1} = \{\alpha_{l,k_l}\}$, then

$$bh_l - a \equiv \nu'(X - Y_l)^{\alpha_{l,k_l}} \pmod{I_{l,k_l}}.$$

Therefore for any $(a, b) \in M_k$, we have

$$bh_l - a \equiv \nu(X - Y_l)^{\alpha_{l,k_l}} \pmod{I_{l,k_l}},$$

and $(a, b) \in M_{k+1}$ if and only if $\nu = 0$.

Define an \mathbb{F} -homomorphism

$$\begin{aligned} \theta &: M_k \longrightarrow \mathbb{F} \\ (a, b) &\longmapsto \nu \end{aligned}$$

Obviously, $\ker(\theta) = M_{k+1}$, $(x_s - y_{l,s})M_k \subseteq M_{k+1}$. We define $H: H((a, b)) = (a, b)$. Then for any $(a, b) \in \mathcal{P}^2$, $H((x_s - y_{l,s})(a, b)) = (x_s - y_{l,s})H((a, b))$. Let $S = M_k$, $S' = M_{k+1}$.

With the definitions above, we can give the form of Fitzpatrick algorithm for multivariate osculatory rational interpolation.

Fitzpatrick algorithm_RI : (Using the minimal Gröbner basis \mathcal{G}_k , compute the minimal Gröbner basis of M_{k+1})

Input: the minimal Gröbner basis $\mathcal{G}_k = \{(a_1, b_1), \dots, (a_{m_k}, b_{m_k})\}$;

Output: the minimal Gröbner basis of M_{k+1} , that is \mathcal{G}_{k+1} ;

Rearrange the elements of \mathcal{G}_k such that $\text{LT}(a_1, b_1) \prec_\xi \dots \prec_\xi \text{LT}(a_{m_k}, b_{m_k})$;
for t from 1 to m_k **do**

$$b_t h_l - a_t \equiv \nu_t(X - Y_l)^{\alpha_{l,k_l}} \pmod{I_{l,k_l}};$$

end do;

if $\nu_t = 0$ for all t **then**

$$\mathcal{G}_{k+1} := \mathcal{G}_k;$$

else

for t from 1 to m_k **do**

Find the least t_k such that $\nu_{t_k} \neq 0$;

end do;

for t from $t_k + 1$ to m_k **do**

$$(a_t, b_t) := (a_t, b_t) - \frac{\nu_t}{\nu_{t_k}}(a_{t_k}, b_{t_k});$$

end do;

$$\mathcal{G}_{k+1} := \left\{ (a_1, b_1), \dots, (a_{t_k-1}, b_{t_k-1}), (a_{t_k}, b_{t_k}) \cdot (x_1 - y_{l,1}), \dots, \right. \\ \left. (a_{t_k}, b_{t_k}) \cdot (x_n - y_{l,n}), (a_{t_k+1}, b_{t_k+1}), \dots, (a_{m_k}, b_{m_k}) \right\}$$

$$\mathcal{G}_{k+1} := \text{minimal Gröbner basis}(\mathcal{G}_{k+1})$$

end if;

return \mathcal{G}_{k+1} ;

We must point out that here we do not require $\mathbf{LC}((a, b)) = 1$ in minimal Gröbner basis.

Example 1. Given the interpolation problem

point	f_i	$\frac{\partial}{\partial x} f$	$\frac{\partial}{\partial y} f$	$\frac{\partial^2}{\partial xy} f$
(-1,2)	2			
(1,1)	3			
(2,1)	4	5	2	
(3,2)	3	4	3	6

Table 1: interpolation

Fix the order \prec_0 , using the Fitzpatrick algorithm_RI, we can compute the minimal Gröbner basis of the submodule M :

$$\begin{aligned} (a_1, b_1) &= \left(\frac{1103}{14528}x^2 - \frac{1367}{14528}xy - \frac{301}{7264}y^2 + \frac{6713}{14528}x - \frac{959}{7264}y - 1, -\frac{61}{908}y^2 + \frac{3047}{14528}x + \right. \\ &\quad \left. \frac{731}{14528}y - \frac{6335}{14528} \right), \\ (a_2, b_2) &= \left(-\frac{19899}{314176}x^2 + \frac{43619}{314176}xy - \frac{1999}{157088}y^2 - \frac{153069}{314176}x + \frac{14059}{157088}y + 1, \frac{122}{4909}xy + \right. \\ &\quad \left. \frac{793}{19636}y^2 - \frac{67507}{314176}x - \frac{19127}{314176}y + \frac{135787}{314176} \right), \\ (a_3, b_3) &= \left(\frac{6973}{371696}x^2 + \frac{61515}{371696}xy - \frac{16223}{185848}y^2 - \frac{18057}{28592}x + \frac{4115}{185848}y + 1, \frac{488}{23231}x^2 + \right. \\ &\quad \left. \frac{61}{1787}xy - \frac{89291}{371696}x - \frac{12399}{371696}y + \frac{141603}{371696} \right), \\ (a_4, b_4) &= \left(-\frac{305}{12438}y^3 - \frac{1519}{24876}x^2 + \frac{2959}{24876}xy + \frac{1481}{12438}y^2 - \frac{10769}{24876}x - \frac{673}{6219}y + \right. \\ &\quad \left. 1, \frac{122}{6219}xy + \frac{61}{1382}y^2 - \frac{4847}{24876}x - \frac{1697}{24876}y + \frac{10027}{24876} \right), \\ (a_5, b_5) &= \left(\frac{49}{988}xy^2 - \frac{85}{494}y^3 - \frac{6}{19}xy + \frac{214}{247}y^2 + \frac{107}{247}x - y - \frac{22}{247}, -\frac{15}{247}xy + \right. \\ &\quad \left. \frac{61}{988}y^2 + \frac{30}{247}x - \frac{12}{247}y - \frac{37}{247} \right), \\ (a_6, b_6) &= \left(\frac{31}{474}x^2y - \frac{11}{158}xy^2 + \frac{37}{237}y^3 - \frac{31}{237}x^2 + \frac{77}{158}xy - y^2 - \frac{55}{79}x + \frac{78}{79}y + \right. \\ &\quad \left. \frac{184}{237}, \frac{1}{6}xy - \frac{55}{474}y^2 - \frac{1}{3}x - \frac{5}{474}y + \frac{115}{237} \right), \\ (a_7, b_7) &= \left(\frac{31}{978}x^3 - \frac{11}{326}x^2y + \frac{37}{489}xy^2 + \frac{12}{163}x^2 - \frac{227}{978}xy - \frac{37}{163}y^2 - \frac{679}{978}x + y + \right. \\ &\quad \left. \frac{92}{163}, \frac{79}{978}x^2 - \frac{55}{978}xy - \frac{176}{489}x + \frac{55}{326}y + \frac{115}{326} \right) \end{aligned}$$

Any weak interpolation (a, b) have the form

$$(a, b) = c_1(a_1, b_1) + \dots + c_7(a_7, b_7),$$

where $c_j \in \mathcal{P}$ ($j = 1, \dots, 7$) are free parameters. Choose c_j properly such that $b(Y_i) \neq 0$, $i = 1, \dots, L$, then we can get the interpolation function

$$\frac{a(X)}{b(X)} = \frac{c_1 a_1 + \dots + c_7 a_7}{c_1 b_1 + \dots + c_7 b_7}.$$

4. Neville-like interpolation

Neville's algorithm is used for polynomial interpolation which was derived by Eric Harold Neville. The algorithm aims at determining the value of the interpolating polynomial at a single point x . [22] also derived a Neville type algorithm for univariable rational interpolation.

In this section, we present a Neville-like algorithm for multivariate Cauchy interpolation based on Fitzpatrick algorithm_RI.

Given a set of L distinct points $\{Y_1, \dots, Y_L\}$, $Y_j \in \mathbb{F}^n$, $j = 1, \dots, L$, and the corresponding values $\{f_1, \dots, f_L\}$, $f_j \in \mathbb{F}$, $j = 1, \dots, L$, we want to determine the interpolating value at the point Y_0 .

In this case, we know that $h_j := f_j$, and $M_k := \{(a, b) \in M_{k-1} : bh_k - a = 0 \bmod I_k\} = \{(a, b) \in M_{k-1} : (bh_k - a)|_{Y_k} = 0\}$, $k = 1, \dots, L$. In Fitzpatrick algorithm_RI, if we can get the values $W(i, j) = (b_i h_j - a_i)|_{Y_j} = \nu_{i,j}$ and $(a_i|_{(x_0, y_0)}, b_i|_{(x_0, y_0)})$ recursively without computing the weak interpolation (a, b) , then we can determine the interpolating value at the point Y_0 . It means that using the present values, we can calculate the new values of $W(i, j)$ and $(a_i|_{(x_0, y_0)}, b_i|_{(x_0, y_0)})$ without computing the weak interpolation (a, b) when a new point is added. Based on this idea we get a Neville-Like algorithm for Cauchy interpolation. For simplicity we will restrict ourselves to the case $n = 2$. Three and higher dimensional cases can be treated similarly.

Fix the order \prec_ξ , and $y \prec x$. Let $\{(a_1, b_1), \dots, (a_{m_k}, b_{m_k})\}$ be the minimal Gröbner basis of M_k . Define $W(i, j) = (b_i h_j - a_i)|_{Y_j}$, $i = 1, \dots, m_k$, $j = 1, \dots, L$, $\vec{W}(i, L+1) = (a_i|_{(x_0, y_0)}, b_i|_{(x_0, y_0)})$, $\vec{W}(i, L+2) = \mathbf{LT}((a_i, b_i))$.

Define $\mathbf{W}_i = (W(i, 1), \dots, W(i, L), \vec{W}(i, L+1), \vec{W}(i, L+2))$.

We know that $\langle (1, 0), (0, 1) \rangle$ is the Gröbner basis of $M_0 = \mathcal{P}^2$.

First, using the Gröbner basis of M_0 , we compute

$$\begin{aligned} W(1, j) &= 0 \cdot h_j - 1 = -1, \quad j = 1, \dots, L; \\ W(2, j) &= 1 \cdot h_j - 0 = f_j, \quad j = 1, \dots, L. \end{aligned}$$

Let

$$\vec{W}(1, L+1) = (1, 0), \quad \vec{W}(1, L+2) = \mathbf{LT}((1, 0)) = (1, 0),$$

$\vec{W}(2, L+1) = (0, 1)$, $\vec{W}(2, L+2) = \mathbf{LT}((0, 1)) = (0, 1)$.
 We will compute the $\mathbf{W}_i^{(k)}$ recursively by $\mathbf{W}_1, \mathbf{W}_2$.
 Define the matrix

$$\mathcal{W}_0 := \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} \triangleq (\mathbf{W}_1, \mathbf{W}_2)^T.$$

Let $\mathcal{W}_{k-1} = (\mathbf{W}_1, \dots, \mathbf{W}_{m_k})^T$. Denote by $\sharp \mathcal{W}_{k-1}$ the number of rows.

Neville-like algorithm:

Input: $\mathcal{W}_0 = (\mathbf{W}_1, \mathbf{W}_2)^T$; L ;

Output: \mathcal{W}_L ;

for k **from** 1 **to** L **do**

$m_k := \sharp \mathcal{W}_{k-1}$;

rearrange the elements of \mathcal{W}_{k-1} so that

$$\mathcal{W}_{k-1} = (\mathbf{W}_1, \dots, \mathbf{W}_{m_k})^T$$

$$\text{and } W(1, L+2) \prec_\xi \dots \prec_\xi W(m_k, L+2);$$

for i **from** 1 **to** m_k **do**

Find the least i_0 such that $W(i_0, k) \neq 0$;

end do;

if $W(i, k) = 0$ **for all** i **then**

$$\mathcal{W}_k := \mathcal{W}_{k-1}$$

else

for i **from** $i_0 + 1$ **to** m_k **do**

for j **from** 1 **to** L **do**

$$W(i, j) := W(i, j) - \frac{W(i, j_0)}{W(i_0, j_0)} W(i_0, j);$$

end do;

$$\vec{W}(i, L+1) := \vec{W}(i, L+1) - \frac{W(i, j_0)}{W(i_0, j_0)} \vec{W}(i_0, L+1);$$

$$\vec{W}(i, L+2) := \vec{W}(i, L+2);$$

end do;

for j **from** 1 **to** L **do**

$$W(m_k + 1, j) := W(i_0, j) \cdot (x_j - x_{j_0});$$

$$W(m_k + 2, j) := W(i_0, j) \cdot (y_j - y_{j_0});$$

end do;

$$\vec{W}(m_k + 1, L+1) := \vec{W}(i_0, L+1) \cdot (x_0 - x_{j_0});$$

$$\vec{W}(m_k + 1, L+2) := \vec{W}(i_0, L+2) \cdot x;$$

$\vec{W}(m_k + 2, L + 1) := \vec{W}(i_0, L + 1) \cdot (y_0 - y_{j_0});$
 $\vec{W}(m_k + 2, L + 2) := \vec{W}(i_0, L + 2) \cdot y;$
 $\mathcal{W}_k := (\mathbf{W}_1, \dots, \mathbf{W}_{i_0-1}, \mathbf{W}_{i_0+1}, \dots, \mathbf{W}_{m_k+2})^T;$
 $\mathcal{W}_k := \text{Minimal Gröbner basis}(\mathcal{W}_k)$
end if;
end do;
return $\mathcal{W}_L;$

Minimal Gröbner basis(\mathcal{W})

Input: $\mathcal{W};$

Output: $\widetilde{\mathcal{W}} = (\mathbf{W}_1, \dots, \mathbf{W}_m)^T$ with no $\vec{W}(i, L + 2)$ is divisible by $\vec{W}(j, L + 2)$ for $i \neq j$.

Example 2. Given objective function $\ln(x^2 + y^2)$, we will use the values at the points $(1.75, 1.75)$, $(2.25, 1.75)$, $(1.75, 2.25)$, $(2.25, 2.25)$, $(1.85, 1.85)$, $(2.15, 1.85)$, $(1.85, 2.15)$, $(2.15, 2.15)$ to estimate the value at the point $(2, 2)$.

Fix the order \prec_0 , $L = 8$, Neville-like algorithm outputs \mathcal{W}_8 :

0.	0.	0.	0.	0.	0.	0.	0.	$(8.200 \times 10^{-7}, 4.100 \times 10^{-7})$	$(x^2, 0)$
0.	0.	0.	0.	0.	0.	0.	0.	$(12.45851102, 5.991319801)$	$(0, y^2)$
0.	0.	0.	0.	0.	0.	0.	0.	$(-37.67095363, -18.11585643)$	$(0, xy)$
0.	0.	0.	0.	0.	0.	0.	0.	$(37.67095324, 18.11585624)$	$(0, x^2)$
0.	0.	0.	0.	0.	0.	0.	0.	$(-1.225410500, -0.5894116890)$	$(y^3, 0)$
0.	0.	0.	0.	0.	0.	0.	0.	$(3.470473588, 1.668910238)$	$(xy^2, 0)$

Table 2: \mathcal{W}_8 for $\ln(x^2 + y^2)$

From the $(8 + 1)$ -th column of \mathcal{W}_8 , we can see that each of the vectors $\vec{W}(i, 8 + 1) = (a_i|_{(2,2)}, b_i|_{(2,2)})$, $i = 1, \dots, 6$, gives an approximate value $\frac{a_i|_{(2,2)}}{b_i|_{(2,2)}}$ of $\ln(2^2 + 2^2)$.

Here we choose

$$\frac{\sum_{i=1}^{m_k} \text{sgn}(b_i|_{Y_0}) \cdot a_i|_{Y_0}}{\sum_{i=1}^{m_k} \text{sgn}(b_i|_{Y_0}) \cdot b_i|_{Y_0}},$$

as our estimation value (see Table 3), where $Y_0 = (2, 2)$, $\text{sgn}(x)$ satisfies: if $x \geq 0$, $\text{sgn}(x) = 1$, else $\text{sgn}(x) = -1$. Actually $\ln(2^2 + 2^2) = 2.079441542$.

i	(x_i, y_i)	$\ln(x_i^2 + y_i^2)$	interpolating value of \mathcal{W}_i
1	(1.75,1.75)	1.812378756	1.312378756
2	(2.25,1.75)	2.094945728	1.812378756
3	(1.75,2.25)	2.094945728	2.122484930
4	(2.25,2.25)	2.315007613	2.107686660
5	(1.85,1.85)	1.923518459	2.082067864
6	(2.15,1.85)	2.085050780	2.082067864
7	(1.85,2.15)	2.085050780	2.079431546
8	(2.15,2.15)	2.224082865	2.079439873

Table 3: estimation value of $\ln(2^2 + 2^2)$

i	Y_i	$\sqrt{1 - (x_i)^2 - (y_i)^2}$	interpolating value of \mathcal{W}_i
1	(0.45,0.45)	0.7713624310	0.6713624310
2	(0.55,0.45)	0.7035623640	0.7078673362
3	(0.45,0.55)	0.7035623640	0.7035623636
4	(0.55,0.55)	0.6284902545	0.7035623639
5	(0.5,0.45)	0.7399324293	0.7035623639
6	(0.5,0.55)	0.6689544080	0.7047928585
7	(0.45,0.5)	0.7399324293	0.7071486038
8	(0.55,0.5)	0.6689544080	0.7071187945

Table 4: estimation value of $\sqrt{1 - (0.5)^2 - (0.5)^2}$

Example 3. Given the values of $\sqrt{1 - x^2 - y^2}$ at the points (0.45,0.5), (0.55,0.45), (0.45,0.55), (0.55,0.55), (0.5,0.45), (0.5,0.55), (0.45,0.55), (0.55,0.5). Fix the order \prec_0 , we estimate the value of $\sqrt{1 - (0.5)^2 - (0.5)^2}$ by

$$\frac{\sum_{i=1}^{m_k} \mathbf{sgn}(b_i |_{(0.5,0.5)}) \cdot a_i |_{(0.5,0.5)}}{\sum_{i=1}^{m_k} \mathbf{sgn}(b_i |_{(0.5,0.5)}) \cdot b_i |_{(0.5,0.5)}} \quad (\text{see Table 4}).$$

$$\text{Actually } \sqrt{1 - (0.5)^2 - (0.5)^2} = 0.7071067812.$$

Example 4. Given the values of $\exp(x^2 + y)$ at the points (2, 2.95), (2, 3.05), (1.95, 3), (2.05, 3), (1.975, 2.975), (1.975, 3.025), (2.025, 2.975), (2.025, 3.025).

Fix the order \prec_0 , we still choose $\frac{\sum_{i=1}^{m_k} \mathbf{sgn}(b_i |_{(2,3)}) \cdot a_i |_{(2,3)}}{\sum_{i=1}^{m_k} \mathbf{sgn}(b_i |_{(2,3)}) \cdot b_i |_{(2,3)}}$ as our estimation value(see Table 5).

$$\text{Actually } \exp(2^2 + 3) = 1096.633158.$$

i	Y_i	$\exp(x_i^2 + y_i)$	interpolating value of \mathcal{W}_i
1	(2, 2.95)	1043.149728	1043.099728
2	(2, 3.05)	1152.858743	1044.131824
3	(1.95, 3)	900.0947180	1043.425504
4	(2.05, 3)	1342.783531	1102.658424
5	(1.975, 2.975)	968.3804142	1097.459656
6	(1.975, 3.025)	1018.030340	1096.945601
7	(2.025, 2.975)	1182.782509	1096.552830
8	(2.025, 3.025)	1243.425065	1096.660126

Table 5: estimation value of $\exp(2^2 + 3)$

5. Conclusion

In this paper, we apply the Fitzpatrick algorithm to osculatory rational interpolation, and get the parametric solution of all the interpolation functions with the given complexity .

For Cauchy interpolation, we present a Neville-like algorithm to determine the value of interpolating function at a single point without computing the rational interpolation function (several points can be treated similarly). Since each of the vectors $\vec{W}(i, L + 1) = (a_i|_{Y_0}, b_i|_{Y_0})$ gives an approximate value, we choose

$$\frac{\sum_{i=1}^{m_k} \text{sgn}(b_i|_{Y_0}) \cdot a_i|_{Y_0}}{\sum_{i=1}^{m_k} \text{sgn}(b_i|_{Y_0}) \cdot b_i|_{Y_0}},$$

as our estimation value. From the examples we can see that the Neville-like algorithm is effective.

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